Feb 9

Ex: $f(x) = \begin{cases} x^2 & \text{when } x \geq 0 \\ 2x & \text{when } x < 0 \end{cases}$

Find $f'(0)$ (possibly undefined)

need to compute

$$f'(x) := \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$f(0) = 0$

$$f'(0) := \lim_{\Delta x \to 0} \frac{f(\Delta x) - \cancel{f(0)}}{\Delta x}$$

$\Delta x = x - 0 = x$

$$= \lim_{x \to 0} \frac{f(x)}{x} \quad \text{= should be} \quad \lim_{x \to 0^-} \boxed{\frac{f(x)}{x}} = \lim_{x \to 0^+} \boxed{\frac{f(x)}{x}}$$

the problem first & third blanks of the question ask for these expression

$$\lim_{x \to 0^-} \frac{f(x)}{x} = \lim_{x \to 0^-} \left(\frac{2x}{x} = 2\right) = 2$$

if not, not the limits do not exist then the final answer is that $f'(0)$ is undefined

Ex. $f = \begin{cases} -6x^2 + 6x & x < 0 \\ 5x^3 - x & x \geq 0 \end{cases}$

Find $f'(0)$:

$\Delta x = x - a = x - 0 = x$

$$\lim_{x \to 0^-} \frac{f(x)}{x} = \lim_{x \to 0^-} \left(\frac{-6x^2 + 6x}{x}\right) = \lim_{x \to 0^-} \boxed{(-6x+6)} = 6$$

$$\lim_{x \to 0^+} \frac{f(x)}{x} = \lim_{x \to 0^+} \left(\frac{5x^3 - x}{x}\right) = \lim_{x \to 0^+} \boxed{(5x^2 - 1)} = -1$$

$$\text{Quotient rule:}\quad \frac{(\sqrt{x})'3^x - \sqrt{x}(3^x)'}{(3^x)^2} = \frac{\frac{1}{2}x^{-\frac{1}{2}}\cdot 3^x - x^{\frac{1}{2}}\cdot 3^x \ln 3}{(3^x)^2} = \frac{\frac{1}{2}x^{-\frac{1}{2}} - x^{\frac{1}{2}}\ln 3}{3^x}$$

$$\text{Product rule:}\quad \left(\sqrt{x}\cdot\left(\frac{1}{3}\right)^x\right)' = \frac{1}{2}x^{-\frac{1}{2}}\left(\frac{1}{3}\right)^x + x^{\frac{1}{2}}\left(\frac{1}{3}\right)^x \ln\frac{1}{3} = \frac{\frac{1}{2}x^{-\frac{1}{2}} - x^{\frac{1}{2}}\ln 3}{3^x}$$

Exercise 4.2.3. Use two different methods to compute $\left(\frac{1-x^2}{\sqrt{x}}\right)'$.

Example 4.2.3. Suppose $f(x)$ and $g(x)$ are differentiable. Given $f(1) = 1$, $f'(1) = 2$, $g(1) = 3$, $g'(1) = 4$. Find the value of

$$\frac{d}{dx}\left(f(x)g(x)\right)$$

at $x = 1$.

Solution. By the product rule

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x).$$

At $x = 1$, the above is

$$\left.\frac{d}{dx}(f\cdot g)\right|_{x=1} = f'(1)g(1) + f(1)g'(1) = 2\times 3 + 1\times 4 = 10.$$

■

Example 4.2.4. Suppose $f(x)$, $g(x)$, $h(x)$ are differentiable. Compute

$$\frac{d}{dx}\left(f(x)g(x)h(x)\right).$$

Solution.

$$\begin{aligned}\frac{d}{dx}\left(f(x)g(x)h(x)\right) &= \left(f(x)g(x)\right)\frac{d}{dx}h(x) + h(x)\frac{d}{dx}\left(f(x)g(x)\right)\\ &= f(x)g(x)h'(x) + h(x)\left(f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)\right)\\ &= f(x)g(x)h'(x) + f(x)g'(x)h(x) + f'(x)g(x)h(x).\end{aligned}$$

■

Or. can write $f(x)\,g(x)\,h(x) = f\cdot(g\cdot h)$

$\rightarrow$ Leibniz rule: $(fgh)' = f'(g\cdot h) + f(g\cdot h)'$ ↑ Leibniz again

$= f'(g\cdot h) + f(g\cdot h' + g'\cdot h)$

4.3 The Chain Rule

(for composite functions / change of variable)

Theorem 6 (The Chain Rule)**.**

If $y = f(u)$ *is a differentiable function of* u,

$u = g(x)$ *is a differentiable function of* x,

$y = f \circ g$

regarded as change of variable

new variable $u = g(x)$

then the composite function $y = f(g(x))$ *is a differentiable function of* x*, and*

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

or equivalently

$$\boxed{\frac{dy}{dx} = f'(g(x))g'(x).}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

How to understand? (Intuition)

Consider the difference quotient, $\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta y}{\Delta u} \cdot \dfrac{\Delta u}{\Delta x}$, take limit as $\Delta x \to 0$.

notation for infinitesimal "Δ"

replace Δ by d when we take the difference Δ to go to 0.

Example 4.3.1. Compute:

$$\frac{d}{dx}(1+2x)^5.$$

Solution. Set $y = f(u) = u^5$ and $u = g(x) = 1 + 2x$. Then $f(g(x)) = (1 + 2x)^5$. $= u^5$

By chain rule,

$$f'(u) = \frac{dy}{du} = 5u^4 \qquad \text{and} \qquad g'(x) = \frac{du}{dx} = 2.$$

$$\frac{du^5}{du} = 5u^4$$

$$\frac{du}{dx} = \frac{d(1+2x)}{dx} = 2$$

Hence

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (5u^4)(2) = 10(1+2x)^4.$$

or

$$\frac{dy}{dx} = f'(g(x))g'(x) = 10(1+2x)^4.$$

■

Example 4.3.2. Compute:

$$y = \frac{d}{dx}\sqrt{1+\sqrt{x}}. = u.$$

Solution. Let $y = f(u) = \sqrt{u}$, $u = g(x) = 1+\sqrt{x}$. Then $f(g(x)) = \sqrt{1+\sqrt{x}}$.

$$\frac{dy}{du} = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}} \quad \text{and} \quad \frac{du}{dx} = \frac{1}{2\sqrt{x}}.$$

$$\frac{dy}{du} = \frac{d\sqrt{u}}{du} = \frac{1}{2}u^{-\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{d}{dx}(1+\sqrt{x}) = \frac{1}{2}x^{-\frac{1}{2}}$$

Therefore

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2\sqrt{u}}\frac{1}{2\sqrt{x}} = \frac{1}{4\sqrt{x}\sqrt{1+\sqrt{x}}}.$$

■

Remark. No need to write the fomulas $f(u)$, $g(x)$ when we are skillful, just remember to differentiate layer by layer: outer ~~than~~ then inner.

For example,

$$\frac{d}{dx}(x+e^x)^{2019} = \underbrace{2019(\overbrace{\boxed{x+e^x}}^{g(x)})^{2018}}_{\text{outer}}\underbrace{(1+e^x)}_{\text{inner}}$$

Example 4.3.3. Using $(e^x)' = e^x$ and the chain rule, we can prove that $(a^x)' = a^x \ln a \quad (a > 0)$.

Proof. Note:

$$\boxed{a^x = e^{\ln a^x}} \quad \text{(Very useful technique!)}$$

$$a = e^{\ln a}.$$

$$a^x = (e^{\ln a})^x = e^{x\ln a}$$

Then,

$$\begin{aligned}(a^x)' &= (e^{\ln a^x})' \\ &= (e^{x\ln a})' \\ &= e^{x\ln a}\cdot \ln a \\ &= a^x \cdot \ln a.\end{aligned}$$

let $u = x \ln a$ constant

$$\frac{du}{dx} = (\ln a)\frac{dx}{dx} = \ln a.$$

□

$$\frac{d}{dx}(a^x) = \frac{de^u}{du}\cdot\frac{du}{dx} = e^u\cdot\ln a = e^{x\ln a}\cdot\ln a = a^x\cdot\ln a$$

Ex: Derive the formula for $\frac{d}{dx}(\log_a x)$ using $\frac{d}{dx}\ln x = \frac{1}{x}$.

Example 4.3.4. Use product rule and chain rule to prove the quotient rule.

Proof. By product rule, we have

$$\left(\frac{f}{g}\right)' = \left(f \cdot \frac{1}{g}\right)' = f' \cdot \frac{1}{g} + f \cdot \left(\frac{1}{g}\right)'.$$

For $\left(\frac{1}{g}\right)'$, let $y = \frac{1}{u}$, $u = g(x)$, then by the chain rule,

$y =$

$$\frac{d}{dx}\left(\frac{1}{g}\right) = \frac{dy}{du} \cdot \frac{du}{dx} = -\frac{1}{g^2(x)} g'(x).$$

chain rule

$$\frac{d\left(\frac{1}{u}\right)}{du} \cdot \frac{d\,g(x)}{dx} = (-1)\,u^{-2}\frac{dg}{dx} = -g^{-2}\frac{dg}{dx}$$

Therefore,

$$\left(\frac{f}{g}\right)' = f'\frac{1}{g} - f\frac{g'}{g^2} = \frac{f'g - fg'}{g^2}.$$

□

Example 4.3.5. Compute

$$\frac{d}{dx} e^{\sqrt{x^2+x}}.$$

Solution.

$$\begin{aligned} \frac{dy}{dx} &= e^{\sqrt{x^2+x}} \cdot (\sqrt{x^2+x})' && \text{(chain rule, } y = e^u, u = \sqrt{x^2+x}) \\ &= e^{\sqrt{x^2+x}} \cdot \frac{1}{2}(x^2+x)^{-\frac{1}{2}} \cdot (2x+1) && \text{(chain rule again, } u = \sqrt{w}, w = x^2+x) \end{aligned}$$

■

Exercise 4.3.1. Prove

1.
$$\frac{d}{dx}(g(x))^n = n(g(x))^{n-1} g'(x).$$

2.
$$\frac{d}{dx} e^{\sqrt{\frac{x-1}{x+1}}} = e^{\sqrt{\frac{x-1}{x+1}}} \cdot (x-1)^{-\frac{1}{2}} \cdot (x+1)^{-\frac{3}{2}}.$$

u

y

let $u = \sqrt{\frac{x-1}{x+1}}$

$y = e^u$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= e^u \cdot \frac{d}{dx}\left(\sqrt{\frac{x-1}{x+1}}\right)$$

$v = \frac{x-1}{x+1}$

$$= e^{\sqrt{\frac{x-1}{x+1}}} \cdot \frac{d}{dx} v^{\frac{1}{2}} = e^{\sqrt{\frac{x-1}{x+1}}} \cdot \frac{dv^{\frac{1}{2}}}{dv}\frac{dv}{dx}$$

chain rule for this.

$\frac{1}{2} v^{-\frac{1}{2}} \cdot \frac{d\left(\frac{x-1}{x+1}\right)}{dx}$

don't forget to re express in terms of x.

apply quotient rule

4.3.1 Technique Using Logarithmic Differentiation

Example 4.3.6. Prove

$$\boxed{\frac{d}{dx}\ln|x| = \frac{1}{x}, \quad x \neq 0.}$$

recall that $\frac{d \ln x}{dx} = \frac{1}{x}$ when $x > 0$

Proof. Let

$$y = \ln|x| = \begin{cases} \ln x, & \text{if } x > 0 \\ \ln(-x), & \text{if } x < 0 \end{cases}$$

For $x > 0$, $\dfrac{dy}{dx} = \dfrac{1}{x}$;

let $u = -x$

For $x < 0$, $\dfrac{dy}{dx} = \dfrac{1}{-x}\cdot(-1) = \dfrac{1}{x}$. (by chain rule)

$$\frac{dy}{dx} = \frac{d\ln u}{du}\frac{du}{dx} = \frac{1}{u}\cdot\frac{d(-x)}{dx} = -\frac{1}{-x} = \frac{1}{x}$$

□

Example 4.3.7. If $y = \sqrt[3]{\dfrac{(x-2)(x-3)^2}{x-5}}$, then find $\dfrac{dy}{dx}$.

Solution.

$$\begin{aligned} y^3 &= \frac{(x-2)(x-3)^2}{x-5} \\ \ln y^3 &= \ln\frac{(x-2)(x-3)^2}{x-5} \\ 3\ln y &= \ln(x-2) + 2\ln(x-3) - \ln(x-5) \\ \frac{3}{y}\frac{dy}{dx} &= \frac{1}{x-2} + \frac{2}{x-3} - \frac{1}{x-5} \\ \frac{dy}{dx} &= \frac{y}{3}\left(\frac{1}{x-2} + \frac{2}{x-3} - \frac{1}{x-5}\right) \\ \frac{dy}{dx} &= \frac{1}{3}\sqrt[3]{\frac{(x-2)(x-3)^2}{x-5}}\left(\frac{1}{x-2} + \frac{2}{x-3} - \frac{1}{x-5}\right) \end{aligned}$$

$\ln(y^a) = a\ln y$

$\ln(a\cdot b) = \ln a + \ln b$

$(x-2)(x-3)^2(x-5)^{-1}$

$u = x-2$

$$\frac{d\ln(x-2)}{dx} = \frac{d\ln u}{du}\frac{d(x-2)}{dx} = \frac{1}{u}\cdot 1 = \frac{1}{x-2}$$

$$\frac{3\,d\ln y}{dx} = 3\frac{d\ln y}{dy}\cdot\frac{dy}{dx} = 3\frac{1}{y}\frac{dy}{dx}$$

■

Alternatively, view y as an implicit function of x defined by the relation

$(x-5)y^3 = (x-2)(x-3)^2$ (Chapter 5)

Example 4.3.8. Compute the derivative of x^x, $\quad x > 0$.

$a^x = e^{(\ln a)x}$

Solution. Write $x^x = e^{x \ln x}$. Let $y = e^u$ and $u = x \ln x$. Then

$= y$

$\frac{da^x}{dx}$ $\quad$ $\frac{dx^a}{dx}$

$$
\begin{aligned}
\frac{d}{dx}x^x &= \frac{dy}{du}\frac{du}{dx} \\
&= e^u(\ln x \frac{dx}{dx} + x\frac{d \ln x}{dx}) \\
&= e^u(\ln x + x\frac{1}{x}) \\
&= x^x(\ln x + 1).
\end{aligned}
$$

$\frac{dy}{dx} = \frac{de^u}{du}\frac{d(x\ln x)}{dx}$

$= e^u$

$e^u = e^{x \ln x} = x^x$

previous results for this doesn't apply here b/c in these formulae a has to be a constant

■

Exercise 4.3.2. Let $y = f(x)^{g(x)}$. Prove that $y' = f(x)^{g(x)}\left(g'(x)\ln f(x) + \frac{f'(x)}{f(x)}g(x)\right)$.

Chapter 5: Differentiation II

Learning Objectives:
(1) Use implicit differentiation to find slope.
(2) Discuss inverse function and its derivatives.
(3) Study the higher order derivative.

5.1 Differentiating Implicit Functions and Inverse Functions

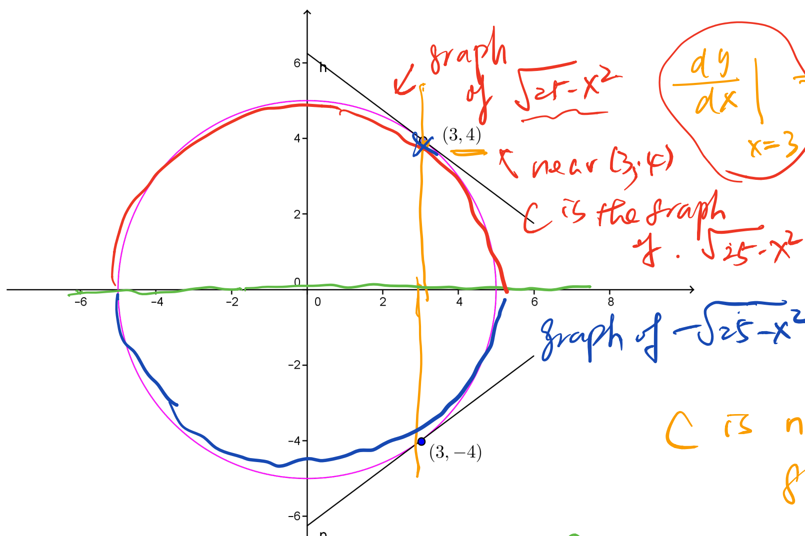
5.1.1 Implicit functions

$C = \{(x,y) \mid x^2+y^2=25\}$

Example 5.1.1. Consider the circle on the $x - y$ plane defined by $x^2 + y^2 = 25$. Find the equation of the tangent line to the circle at $(3, 4)$.

$\in C$. b/c $3^2+4^2=25$.

Solution. **Method 1. Express y in terms of x explicitly.**



$\left.\frac{dy}{dx}\right|_{x=3}$ = slope of the tangent line to C at $(3,4)$

C is not the graph of a function (vertical line test fails)

express C as the union of the graphs of the two functions $y=\sqrt{25-x^2}$ and $y=-\sqrt{25-x^2}$

$$x^2 + y^2 = 25 \quad \Rightarrow y = \pm\sqrt{25 - x^2},$$

Restrict to a small neighbourhood of the point $(3, 4)$ on the curve, $y > 0$ can be uniquely given by $y = \sqrt{25 - x^2}$.

$y = \sqrt{25-x^2} = \sqrt{u}$

$u = 25 - x^2$

slope of this line is given by $\left.\frac{dy}{dx}\right|_{x=3}$

So,

$$y' = -\frac{x}{\sqrt{25-x^2}}$$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

when $x = 3$, $y' = -\frac{3}{4}$. The equation of the tangent line to the curve at $(3, 4)$ is

$$y - 4 = -\frac{3}{4}(x-3),$$
$$y = -\frac{3}{4}x + \frac{25}{4}.$$

actually y might not be expressed as a function of x

Method 2. Implicit differentiation. think of y as defined "implicitly" by

Regard y as a function $y(x)$ without explicit formula. Differentiate both sides of $x^2+y^2 = 25$ with respect to x, and then solve algebraically for $\frac{dy}{dx}$.

$\frac{d}{dx}(y^2+x^2) = \frac{d}{dx}(25)$

"regarding y as a function of x"

$$2x + \frac{d}{dx}(y^2) = 0$$
$$2x + 2y\frac{dy}{dx} = 0 \quad \text{(chain rule)}$$
$$\frac{dy}{dx} = -\frac{x}{y}$$

$\frac{dy^2}{dx} = \frac{dy^2}{dy}\frac{dy}{dx} = 2y\frac{dy}{dx}$

So,

$$\left.\frac{dy}{dx}\right|_{(3,4)} = -\frac{3}{4}.$$

can directly plug in the values of $(x, y) = (3, 4)$ without finding the explicit formula of y as a function of x

Then, find the tangent line in the same way as with Method 1.

■

Remark. Method 2 is referred to as implicit differentiation, which is very useful to compute derivatives of functions not defined by explicit formulae.

Example 5.1.2. Let $y = f(x)$ be a differentiable function of x that satisfies the equation $x^2y + y^2 = x^3$. Find the derivative $\frac{dy}{dx}$.

take $\frac{d}{dx}$ on both sides

Solution. You are going to differentiate both sides of the given equation with respect to x. So that you will not forget that y is actually a function of x, temporarily use the alternative notation $f(x)$ for y, and begin by rewriting the equation as

$$x^2f(x) + (f(x))^2 = x^3.$$

$\frac{d}{dx}(x^2y + y^2) = \frac{d}{dx}x^3$

$2x \cdot y + x^2\frac{dy}{dx} + 2y\frac{dy}{dx} = 3x^2 \rightsquigarrow (x^2+2y)\frac{dy}{dx} = 3x^2 - 2xy$

Now differentiate both sides of this equation term by term with respect to x:

$$\begin{aligned} \frac{d}{dx}[x^2 f(x) + (f(x))^2] &= \frac{d}{dx}[x^3] \\ \rightsquigarrow \left[x^2\frac{df}{dx} + f(x)\frac{d}{dx}(x^2)\right] + 2f(x)\frac{df}{dx} &= 3x^2. \end{aligned} \tag{5.1}$$

Thus, we have

$$\begin{aligned} x^2\frac{df}{dx} + f(x)(2x) + 2f(x)\frac{df}{dx} &= 3x^2 \\ \rightsquigarrow [x^2 + 2f(x)]\frac{df}{dx} &= 3x^2 - 2xf(x) \\ \rightsquigarrow \frac{dy}{dx} &= \frac{3x^2 - 2xf(x)}{x^2 + 2f(x)}. \end{aligned} \tag{5.2}$$

Finally, replace $f(x)$ by y to get

$$\frac{dy}{dx} = \frac{3x^2 - 2xy}{x^2 + 2y}.$$

∎

Summary: Carrying out Implicit Differentiation

Suppose an equation defines y implicitly as a differentiable function of x. To find $\frac{dy}{dx}$:

1. Differentiate both sides of the equation with respect to x. Remember that y is really a function of x, and use the chain rule when differentiating terms containing y.
2. Solve the differentiated equation algebraically for $\frac{dy}{dx}$ in terms of x and y.

Example 5.1.3. Consider the curve defined by

$$x^3 + y^3 = 9xy.$$

1. Compute $\frac{dy}{dx}$. as a function of both x and y
2. Find the slope of the tangent line to the curve at $(4, 2)$.

Solution. Starting with

$$x^3 + y^3 = 9xy,$$

we apply the differential operator $\frac{d}{dx}$ to both sides of the equation to obtain

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}9xy.$$

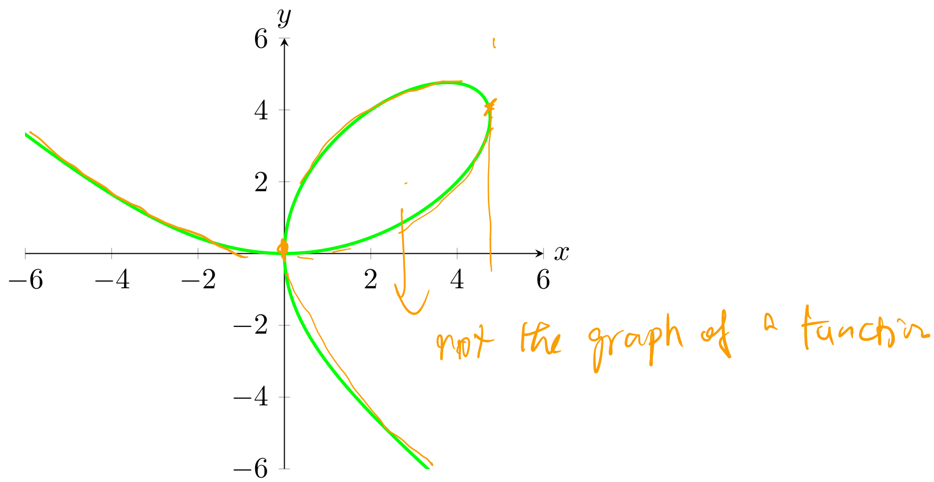


Figure 5.1: A plot of $x^3 + y^3 = 9xy$. While this is not a function of y in terms of x, the equation still defines a relation between x and y.

Applying the sum rule, we see that

$$\frac{d}{dx}x^3 + \frac{d}{dx}y^3 = \frac{d}{dx}9xy.$$

Let's examine each of the terms above in turn. To begin,

$$\frac{d}{dx}x^3 = 3x^2.$$

On the other hand, $\frac{d}{dx}y^3$ is treated somewhat differently. Here, viewing $y = y(x)$ as an implicit function of x, we have by the chain rule that

$$\begin{aligned}\frac{d}{dx}y^3 &= \frac{d}{dx}(y(x))^3 \\ &= 3(y(x))^2 \cdot y'(x) \\ &= 3y^2\frac{dy}{dx}.\end{aligned}$$

Consider the final term $\frac{d}{dx}(9xy)$. Regarding $y = y(x)$ again as an implicit function, we have:

$$\begin{aligned}\frac{d}{dx}(9xy) &= 9\frac{d}{dx}\big(x \cdot y(x)\big) \\ &= 9\left(x \cdot y'(x) + y(x)\right) \\ &= 9x\frac{dy}{dx} + 9y.\end{aligned}$$

Putting all the above together, we get:

$$3x^2 + 3y^2\frac{dy}{dx} = 9x\frac{dy}{dx} + 9y.$$

Now we solve the preceding equation for $\frac{dy}{dx}$. Write

$$\begin{aligned} & 3x^2 + 3y^2\frac{dy}{dx} = 9x\frac{dy}{dx} + 9y \\ \iff \quad & 3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} = 9y - 3x^2 \\ \iff \quad & \frac{dy}{dx}\left(3y^2 - 9x\right) = 9y - 3x^2 \\ \iff \quad & \frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}. \end{aligned}$$

For the second part of the problem, we simply plug in $x = 4$ and $y = 2$ to the last formula above to conclude that the slope of the tangent line to the curve at $(4, 2)$ is $\frac{5}{4}$. See Figure 5.2. ■

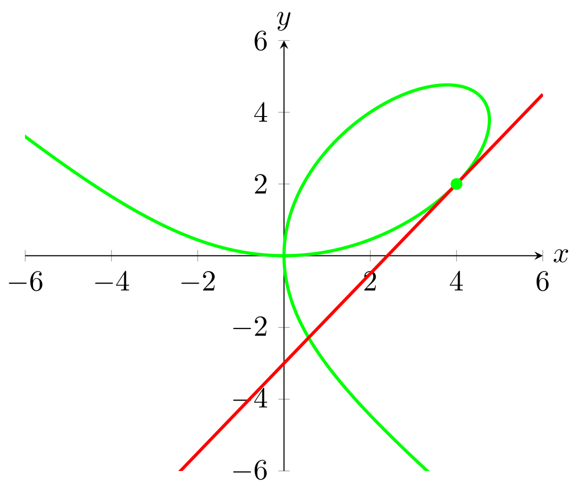


Figure 5.2: A plot of $x^3 + y^3 = 9xy$ along with the tangent line at $(4, 2)$.

Example 5.1.4. Let L be the curve in the $x - y$ plane defined by $x^2 + y^2 + e^{xy} = 2$. Use L to implicitly define a function $y = y(x)$. Find $y'(x)$ at $x = 1$ and the tangent line to the curve L at $(1, 0)$.

Solution. (Note: In this case, there is no good explicit formula for the function $y(x)$.) Differentiate the equation $x^2 + y^2 + e^{xy} = 2$ on both sides with respect to x. We get:

$$\begin{aligned} & 2x + 2yy' + e^{xy}(y + xy') = 0, \\ & \qquad \rightsquigarrow y' = -\frac{2x + e^{xy}y}{2y + e^{xy}x}. \end{aligned}$$